The Decomposability of Normal Distribution

Ziyuan Feng

December 12, 2017

1 Proposition

If $X_1, X_2 \ldots X_n$ are independent random variables, and $X_1 + X_2 + \cdots + X_n$ is normally distributed, then $X_1, X_2 \ldots X_n$ are all normally distributed.

2 Lemma One

Assume $\alpha > 0$. $P(x)$ is a function of x and $F(x)$ is one of the primitive function of $P(x)$. There holds

$$\int_0^{+\infty} x^\alpha P(x)dx = \alpha \int_0^{+\infty} x^{\alpha-1}[1 - F(x)]dx \tag{1}$$

Proof Assume that $b < +\infty$ and the integration converges.

$$\begin{aligned}
\int_0^{b+} x^\alpha P(x)dx &= x^\alpha F(x)|_0^{b+} - \int_0^b \alpha x^{\alpha-1} F(x)dx & (2)\\
&= b^\alpha F(b+) + \alpha \int_0^b x^{\alpha-1}[1 - F(x)] - \alpha \int_0^b x^{\alpha-1} dx & (3)\\
&= b^\alpha [F(b+) - 1] + \alpha \int_0^b x^\alpha [1 - F(x)]dx & (4)
\end{aligned}$$

Let $b \to +\infty$. $b^\alpha[F(b+) - 1] \to 0$. So (1) is proved.

3 Lemma Two

[1] Assume that X is the random variable of an unknown probability distribution D with $p(x)$ as its PDF and $F(x)$ as its CDF. D is a normal distribution if and only if X's characteristic function $\phi(t) \neq 0$, $\forall t$ and there exists a real-value number $0 < \eta < \frac{\sqrt{2}}{2}$ which admits that

$$A(\eta) = \int_{-\infty}^{+\infty} e^{\eta^2 x^2} p(x)dx < \infty \tag{5}$$

Proof

1. **Sufficiency**

   Assume that D is a standard normal distribution, $p(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx$. Its characteristic function is $\phi(t) = e^{\frac{t^2}{2}}$. Obviously, $\phi(t) \neq 0, \forall t$.

$$A(\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\eta^2 x^2} e^{-\frac{x^2}{2}} dx \tag{6}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{(\eta^2 - \frac{1}{2})x^2} dx \tag{7}$$

$$[\int_{-\infty}^{+\infty} e^{(\eta^2 - \frac{1}{2})x^2} dx]^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{(\eta^2 - \frac{1}{2})(x^2+y^2)} dxdy \tag{8}$$

$$(x = r\cos\theta, y = r\sin\theta) = \int_{0}^{+\infty} \int_{0}^{2\pi} e^{(\eta^2 - \frac{1}{2})r^2} rdrd\theta \tag{9}$$

$$= 2\pi \int_{0}^{+\infty} e^{(\eta^2 - \frac{1}{2})r^2} rdr \tag{10}$$

   For the integration to converge, $\eta^2 < \frac{1}{2}$ is needed.Therefore

$$A(\eta) = \sqrt{\frac{1}{1 - 2\eta^2}} < \infty \tag{11}$$

2. **Necessity**

   Assume that $t \in R$.

$$|itx| \leq \eta^2 x^2 + \frac{|ti|^2}{\eta^2} = \eta^2 x^2 + \frac{t^2}{\eta^2} \tag{12}$$

   So the characteristic function ϕ always exists.

$$|\phi(t)| \leq |\int_{-\infty}^{+\infty} e^{\eta^2 x^2 + \frac{t^2}{\eta^2}} p(x)dx| \tag{13}$$

$$= e^{\frac{t^2}{\eta^2}} \int_{-\infty}^{+\infty} e^{\eta^2 x^2} p(x)dx \tag{14}$$

$$= e^{\frac{t^2}{\eta^2}} A(\eta) \tag{15}$$

   Since $\phi(t)$ is analytic at all finite points of the complex plane, it is an entire function. The order ρ of an entire function is defined as

$$\rho = \limsup_{r\to\infty} \frac{\ln(\ln(||f||_{\infty,D_r}))}{\ln r}$$

   where $||f||_{\infty,D_r} = \sup\{|f(x)||x \in D_r\}$. r is the radius of the disk D_r.

According to Hardamand Factorization Theorm[2], an entire function of finite order ρ admits such a factorization that

$$\phi(t) = t^m e^{g(t)} \prod_{n=1}^{\infty} E_{[\rho]}(\frac{z}{a_n})$$

where $g(t)$ is a polynomial item with degree $q \leq \rho$ and $\{a_n\}$ is the non-zero zeros of $\phi(t)$. (15) shows that the degree ρ of $\phi(t)$ is no more than 2. Since $\phi(t) \neq 0$, $\forall t$, hence

$$\phi(t) = e^{at^2+bit} \quad a, b \in C$$

Considering $-i\phi'(0)$ is the expectation and $-\phi''(0)$ is the second order moment, a and b must be real numbers.

Therefore, the characteristic function $\phi(t)$ is the one corresponding to the normal distribution. So D is a normal distribution.

4 Proof

Assume the middle numbers of X_1 and X_2 are zero. Denote that $Y = X_1 + X_2$.

$$\begin{aligned} P(|Y| > t) = P(|X_1 + X_2| > t) &\geq P(\{|X_1| \geq t\} \cap \{|X_2| \geq 0\}) && (16) \\ &= P(|X_1| \geq t)P(|X_2| \geq 0) && (17) \\ &= \frac{1}{2}P(|X_1| > t) && (18) \end{aligned}$$

Therefore

$$1 - F_Y(t) + F_Y(-t) \geq \frac{1}{2}[1 - F_{X1}(t) + F_{X1}(-t)] \tag{19}$$

As for X_1, there exists $\eta > 0$ which permits that

$$
\begin{aligned}
A_{X_1}(\eta) &= \int_{-\infty}^{+\infty} e^{\eta^2 x^2} p(x) dx && (20)\\
&= \int_{0}^{+\infty} e^{\eta^2 x^2} [p(x) + p(-x)] dx && (21)\\
(p(e^{x^2}) = \frac{p(x)}{2xe^{x^2}}) &= \int_{0}^{+\infty} (e^{x^2})^{\eta^2} [p(e^{x^2}) - p(-e^{x^2})] de^{x^2} && (22)\\
(t = e^{x^2}) &= \int_{1}^{+\infty} t^{\eta^2} [p(t) - p(-t)] dt && (23)\\
&= \int_{1}^{+\infty} t^{\eta^2} p(t) dt - \int_{1}^{+\infty} t^{\eta^2} p(-t) dt && (24)\\
(LemmaOne) &= \eta^2 \int_{1}^{+\infty} t^{\eta^2 - 1} [1 - F_{X1}(t)] dt - \eta^2 \int_{1}^{+\infty} t^{\eta^2 - 1} [1 - (1 - F_{X1}(-t))] dt && (25)\\
&= \eta^2 \int_{1}^{+\infty} t^{\eta^2 - 1} [1 - F_{X1}(t) - F_{X1}(-t)] dt && (26)\\
&\leq \eta^2 \int_{1}^{+\infty} t^{\eta^2 - 1} [1 - F_{X_1}(t) + F_{X_1}(-t)] dt && (27)\\
(19) &\leq 2\eta^2 \int_{1}^{+\infty} t^{\eta^2 - 1} [1 - F_Y(t) + F_Y(-t)] dt && (28)\\
(LemmaOne) &= 2 \int_{1}^{+\infty} t^{\eta^2} [p_Y(t) + p_Y(-t)] dt && (29)\\
&\leq 2 \int_{1}^{+\infty} e^{\eta^2 y^2} [p_Y(y) + p_Y(-y)] dy && (30)\\
&= 2A_Y(\eta) < \infty && (31)
\end{aligned}
$$

Obviously, the characteristic function of X_1 is $\phi(t) = \int_{-\infty}^{\infty} e^{itx} dx > 0$, which has no zeros. According to Lemma Two, X_1 is normally distributed.

Since $X_1 + X_2 + \cdots + X_n$ is normally distributed, X_1 and $X_2 + \cdots + X_n$ are independent, both X_1 and $X_2 + \cdots + X_n$ are normally distributed. Continuing this process leads to the conclusion that $X_1, X_2 \ldots X_n$ are all normally distributed.

References

[1] W. Feller. *An Introduction to Probability Theory and Its Applications*. John Wiley Sons, Inc, 1971.

[2] E. W. Weisstein. Hadamard factorization theorem. URL `http://mathworld.wolfram.com/HadamardFactorizationTheorem.html`.